Double Fourier Cosine Series Method for the Flexural Analysis of Kirchhoff Plates on Winkler Foundations

Charles Chinwuba Ike
Department of Civil Engineering, Enugu State University of Science & Technology, Enugu State, Nigeria
ikecc2007@yahoo.com, charles.ike@esut.edu.ng

Abstract

This study presents the double Fourier cosine series method for solving the flexural problem of Kirchhoff plates resting on an elastic foundation of the Winkler type. The problem is a boundary value problem represented by a fourth order partial differential quantum. For the case of simply supported edges, the Dirichlet boundary conditions are identically satisfied by double Fourier cosine series if the plate centre becomes the origin of the Cartesian coordinates. A Fourier cosine series assumption for the unknown deflection function and the known load distribution results in an algebraic problem for the unknown Fourier parameters of the series; which is solved to obtain the deflection function. The paper presents general solutions for deflection and bending moments for arbitrary transverse load distribution and specific solutions for the deflections and bending moments for the specific cases of point load at arbitrary point, and at the centre, uniformly distributed load over the entire plate and sinusoidal load. It was found that the solutions obtained in this study were exact solutions and this was because the double Fourier cosine series used for the deflection shape functions were exact shape functions that satisfies all the Dirichlet boundary conditions. Furthermore, the trial solution was made to satisfy the boundary value problem at all points in the solution domain.

Keywords: Double Fourier cosine series, Kirchhoff plate, Winkler foundation, Boundary value problem.

1. Introduction

1.1 Background

The flexural problems of plates resting on elastic foundations and subjected to transverse loads which are either distributed or acting singularly at points on the plate are frequently encountered in foundation and structural engineering. They occur in the analysis and design of footings. They also arise in the solutions of mathematical problems involving structural problems governed by analogous fourth order partial differential equations of the Kirchhoff plate on Winkler foundation problem. Such problem commonly called soil – structure interaction problems are formulated in mathematical terms by accounting for the soil reaction model of the foundation in the governing differential equations of the plate. [1- 4]. The formulation of the governing equations are usually done using either an equilibrium method or approach or a variational method [1]. In the equilibrium method, the requirements of stress – strain laws, strain displacement relations, the differential equations of equilibrium and the compatibility relations are used in an integral manner to determine the equations of the plate on elastic foundation problem. In the variational method, the problem is formulated in integral form using the principles of the calculus of variations. The total potential energy functional is determined, and variational techniques are applied for solution.

1.2 Plate theories


Plate theories are generally classified as: small deflection thin plate theories, large deflection thin plate theories, moderately thick plate theories and thick plate theories. This study adopts the classical thin plate theory also called the Kirchhoff plate theory or the Kirchhoff – Love plate theory. The Kirchhoff plate theory (KPT) was originally formulated for thin plates with small deflection, and the ratio of the plate thickness to the least governing span, $a$ is less than $1/20 \left( i.e. \frac{h}{a} < \frac{1}{20} \right)$. The basic hypotheses of the KPT include:

(i) straight lines that are orthogonal to the neutral surface (middle plane) of the plate before flexure remain straight after flexural deformation.
(ii) straight lines orthogonal (normal) to the plate’s middle (neutral) surface before bending deformation remain orthogonal to the plate’s middle surface after bending deformation.
(iii) The thickness of the plate is constant and unchanging during flexural deformation.

The Kirchhoff thin plate theory is thus a two-dimensional (2D) approximation of the classical mathematical theory of elasticity in three dimensional (3D) space applied to the formulation of the plate bending problem. It is also considered a two-dimensional extension of the one-dimensional Euler – Bernoulli beam theory. The KPT assumes basically that a middle plane surface which lies at $z = 0$ in between the top and bottom surfaces and is considered neutral during bending deformations can be used to represent the 3D plate in a 2D domain [10]. The KPT like all other two-dimensional plate theories determines the governing equation in terms of unknown deflection of the middle surface, and equations
for stresses, strains, displacements and stress resultants in terms of the applied loads and support conditions. The advantages of the KPT adopted include:

(i) the simplification of the mathematical formulation of the plate problem to a 2D problem in the in-plane coordinates of the plate domain.
(ii) the uncoupling of the flexural and stretching behaviours.
(iii) the resulting governing equation for the plate domain is linear and can be solved using mathematical methods for solving linear partial differential equations.
(iv) stresses can be obtained from the displacements fields using the stress-displacement relations, and strains can be obtained using the generalised Hooke’s law.
(v) it is applicable to thin plate bending problems.

The most significant limitation of the KPT is its disregard of the transverse shear deformation in the formulation, rendering the resulting governing equations unsuitable for the flexural analysis of moderately thick and thick plate problems where transverse shear deformation effects would be significant.

1.3 Foundation (Soil Interaction) models

The soil interaction on the interfacing footing is represented by the soil reactive pressure distribution on the foundation [11]. The soil reaction is expressed using mathematical expressions, which depend on the type of foundation model used. In general, elastic foundation models are categorized as elastic continuum models, simplified elastic continuum models and discrete foundation models [12, 13].

Elastic continuum foundation models idealize the foundation as an elastic continuum and hence use the mathematical theory of elasticity in three-dimensional space coordinate variables as the theoretical framework to derive complicated mathematical expressions for the soil reactions on the interfacing foundation structure. Simplified elastic continuum foundation models rely on the use of simplifying assumptions of stresses and/or deformations to obtain less rigorous and less complex solutions from the elasticity theory for the problem.

Discrete or lumped parameter models are based on the discretization of the elastic foundation parameters and replacing them with a set of closely spaced discrete elastic springs that may or may not be made to interact with one another. The discrete or lumped parameter foundation models commonly found in the literature of soil structure interaction are:

(iii) Pasternak [16] foundation model.
(v) Kerr [18, 19] foundation model.
(vi) Vlasov foundation model [20–24].
(vii) Generalized two-parameter foundation model.
(viii) Generalised n-parameter foundation model.

The simplest model that describes the soil – reaction pressure distribution is the Winkler foundation model. The Winkler model assumes the soil can be replaced by a mechanical analogy of a bed of closely spaced linearly elastic mechanical springs which are not connected to one another and are placed directly beneath the interfacing foundation. It further assumes, that at any point under the foundation the soil reaction pressure \( p(x, y) \) is directly proportional to the deflection \( w(x, y) \) and the proportionality constant \( k_1 \) which is the elastic property of the mechanical spring analogue is the elastic foundation parameter characterising the elastic soil. The Winkler model is represented mathematically by the simple equation:

\[
p(x, y) = k_1 w(x, y) \tag{1}
\]

The proportionality constant \( k_1 \) is the Winkler coefficient or the coefficient of subgrade reaction. The shortcomings of the Winkler model are:

(i) the discontinuity in the deformation at the plate edges is inconsistent with the elastic behaviour of soil as a 3D continuum.
(ii) the independence of the vertical deformation at any point of the vertical deformation of other adjoining/neighbouring points contradict the results of stress analysis from the theory of 3D elasticity.

Despite these limitations, the simple nature of the equation for soil reaction pressure which results in relatively simple equations for the soil – structure interaction (SSI) problem has accounted for its extensive application in describing SSI problems. The Hetenyi, Pasternak, Filonenko – Borodich, Vlasov and generalised two parameter discrete foundation models were formulated to incorporate couplings and interactions of the spring elements, thus introducing second parameters that describe the soil reaction pressure; and are called two parameter discrete foundation models. The soil reaction pressure in two parameter discrete foundation models is given generally by:

\[
p(x, y) = k_1 w(x, y) - k_2 \nabla^2 w(x, y) \tag{2}
\]

where \( k_1 \) is the first discrete parameter, \( k_2 \) is the second discrete parameter and \( \nabla^2 \) is the Laplacian operator, a partial differential operator in two dimensional Cartesian coordinates expressed by:

\[
\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \tag{3}
\]

1.4 Research aim and objectives

The research aim is to use the double Fourier cosine series method to solve the problem of flexural analysis of simply supported Kirchhoff plates resting on Winkler foundations for arbitrary and specific transverse load distributions. The specific objectives are:

(i) to use the double Fourier cosine series method to obtain the general solution for deflections, and bending moments for the flexural problem of simply supported Kirchhoff plate resting on Winkler foundation for the case of general arbitrary distribution of transverse load.
(ii) to transform the boundary value problem of Kirchhoff plate resting on Winkler foundation under arbitrary transverse load distribution to an algebraic problem using the double Fourier cosine series method.
(iii) to solve the resulting algebraic equation, and hence find solutions for the general case of arbitrary distributed transverse load on the Kirchhoff plate on Winkler foundation.
(iv) to obtain solutions for the deflection function and bending moment expressions for the flexural problem of Kirchhoff plates resting on Winkler foundations for the following specific types of transverse load:
(a) point load acting at a given point on the plate domain.
(b) sinusoidal distribution of load over the entire plate domain.
(c) uniform distribution of transverse load on the entire plate domain.

2. Theoretical framework

A rectangular Kirchhoff plate with inplane dimensions of length \( a \), and width \( b \) resting on an elastic foundation of the Winkler type as shown in Figure 1, was considered in the work.

![Rectangular Kirchhoff plate on Winkler foundation under arbitrary (general) load distribution](image)

The governing partial differential equation (PDE) is the fourth order equation defined over the plate domain:

\[
D \nabla^4 w(x, y) + k w(x, y) = p(x, y) \quad (4)
\]

or \( \nabla^4 w(x, y) + \frac{k}{D} w(x, y) = \frac{p(x, y)}{D} \quad (5) \)

where \( -\frac{a}{2} \leq x \leq \frac{a}{2}, -\frac{b}{2} \leq y \leq \frac{b}{2} \)

\[
\nabla^4 = \nabla^2 \nabla^2 = \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)^2 = \frac{\partial^4}{\partial x^4} + 2 \frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4} \quad (6)
\]

\( D \) is the flexural rigidity of the plate material, \( E \) is the Young’s modulus of elasticity, \( k \) is the Winkler modulus of soil reaction, \( w(x, y) \) is the transverse deflection of the plate’s middle surface, \( p(x, y) \) is the distributed transverse load intensity, and \( x \) and \( y \) are the in-plane Cartesian coordinate variables. \( D \) is related to the plate’s elastic and geometrical properties as:

\[
D = \frac{Eh^3}{12(1 - \mu^2)} \quad (7)
\]

where \( h \) is the thickness of the plate and \( \mu \) is the Poisson’s ratio of the plate material.

The origin, \( O \), of the Cartesian coordinates system is chosen at the plate centre and this takes advantage of the symmetrical nature of the thin plate. The definition of the origin at the centre can also be of advantage when the load distribution considered is also symmetrical about the plate centre. For the case of Kirchhoff plate resting on elastic foundation of the Winkler type, with all four edges \( x = \pm a/2, y = \pm b/2 \) simply supported, the geometric and force boundary conditions are as follows:

at \( x = \pm \frac{a}{2} \)

\[
w \left( x = \pm \frac{a}{2}, y \right) = 0 \quad (8)
\]

\[
\frac{\partial^2 w}{\partial x^2} \left( x = \pm \frac{a}{2}, y \right) = 0 \quad (9)
\]

at \( y = \pm \frac{b}{2} \)

\[
w \left( x, y = \pm \frac{b}{2} \right) = 0 \quad (10)
\]

\[
\frac{\partial^2 w}{\partial y^2} \left( x, y = \pm \frac{b}{2} \right) = 0 \quad (11)
\]

3. Methodology

In the double Fourier cosine series method, the unknown function \( w(x, y) \) in the governing PDE as well as the known distribution of load \( p(x, y) \) are assumed in the form of double Fourier cosine series of infinite terms. Thus,

\[
w(x, y) = \sum_{m}^{\infty} \sum_{n}^{\infty} w_{mn} \frac{\bar{m} \pi x}{a} \cos \frac{\bar{m} \pi y}{b} \quad (12)
\]

\[
p(x, y) = \sum_{m}^{\infty} \sum_{n}^{\infty} p_{mn} \frac{\bar{m} \pi x}{a} \cos \frac{\bar{n} \pi y}{b} \quad (13)
\]

\( w_{mn} \) are the unknown generalised deflection parameters

\( p_{mn} \) are the parameters of the Fourier cosine series of the load where \( \bar{m} = 1, 3, 5, 7, \ldots \quad \bar{n} = 1, 3, 5, 7, \ldots \)

It is observed that this assumption for \( w(x, y) \) satisfies all the geometric and force boundary conditions associated with the four edges.

By Fourier series theory, the Fourier series coefficients of the load are:

\[
p_{mn} = \frac{4}{ab} \int_{-a/2}^{a/2} \int_{-b/2}^{b/2} p(x, y) \cos \frac{\bar{m} \pi x}{a} \cos \frac{\bar{n} \pi y}{b} \, dx \, dy \quad (14)
\]

Then the fourth order PDE Equation (5) becomes:

\[
\nabla^4 \sum_{m}^{\infty} \sum_{n}^{\infty} w_{mn} \frac{\bar{m} \pi x}{a} \cos \frac{\bar{m} \pi y}{b} + \frac{k}{D} \sum_{m}^{\infty} \sum_{n}^{\infty} w_{mn} \frac{\bar{m} \pi x}{a} \cos \frac{\bar{n} \pi y}{b} = \frac{1}{D} \sum_{m}^{\infty} \sum_{n}^{\infty} p_{mn} \frac{\bar{m} \pi x}{a} \cos \frac{\bar{n} \pi y}{b} \quad (15)
\]

\[
\sum_{m}^{\infty} \sum_{n}^{\infty} \sum_{m}^{\infty} \sum_{n}^{\infty} w_{mn} \frac{\bar{m} \pi x}{a} \cos \frac{\bar{m} \pi y}{b} \quad (16)
\]

\[
\sum_{m}^{\infty} \sum_{n}^{\infty} \sum_{m}^{\infty} \sum_{n}^{\infty} \frac{\bar{m} \pi x}{a} \cos \frac{\bar{n} \pi y}{b} \quad (17)
\]

\[
\sum_{m}^{\infty} \sum_{n}^{\infty} \sum_{m}^{\infty} \sum_{n}^{\infty} \frac{\bar{m} \pi x}{a} \cos \frac{\bar{n} \pi y}{b} \quad (18)
\]
\[
\sum_{m} \sum_{n} \left( \left( \frac{m \pi}{a} \right)^2 + \left( \frac{n \pi}{b} \right)^2 \right) k w_{mn} \cos \frac{m \pi x}{a} \cos \frac{n \pi y}{b} = \sum_{m} \sum_{n} p_{mn} \cos \frac{m \pi x}{a} \cos \frac{n \pi y}{b} \quad (19)
\]

Multiplying both sides of Equation (19) by \( \cos \frac{m \pi x}{a} \cos \frac{n \pi y}{b} \) and integrating over the plate domain, we have

\[
\int_{-a/2}^{a/2} \int_{-b/2}^{b/2} \sum_{m} \sum_{n} \left( \left( \frac{m \pi}{a} \right)^2 + \left( \frac{n \pi}{b} \right)^2 \right) k w_{mn} \cos \frac{m \pi x}{a} \cos \frac{n \pi y}{b} \cos \frac{m \pi x}{a} \cos \frac{n \pi y}{b} \, dx \, dy = \sum_{m} \sum_{n} p_{mn} \int_{-a/2}^{a/2} \int_{-b/2}^{b/2} \cos \frac{m \pi x}{a} \cos \frac{n \pi y}{b} \cos \frac{m \pi x}{a} \cos \frac{n \pi y}{b} \, dx \, dy \quad \ldots (20)
\]

\[
\sum_{m} \sum_{n} \left( \left( \frac{m \pi}{a} \right)^2 + \left( \frac{n \pi}{b} \right)^2 \right) k w_{mn} = \sum_{m} \sum_{n} p_{mn} \quad \ldots (21)
\]

Using the orthogonality properties of the trigonometric cosine series,

\[
\sum_{m} \sum_{n} \left( \left( \frac{m \pi}{a} \right)^2 + \left( \frac{n \pi}{b} \right)^2 \right) k w_{mn} = \sum_{m} \sum_{n} p_{mn} \quad \ldots (22)
\]

\[
\sum_{m} \sum_{n} \left( \left( \frac{m \pi}{a} \right)^2 + \left( \frac{n \pi}{b} \right)^2 \right) k w_{mn} = \sum_{m} \sum_{n} \frac{p_{mn}}{D} \quad \ldots (23)
\]

\[
w_{mn} = \frac{p_{mn}}{D} \quad \ldots (24)
\]

\[
w_{mn} = \frac{p_{mn}}{D} \quad \ldots (25)
\]

\[
w_{mn} = \frac{p_{mn}}{D} \quad \ldots (26)
\]

Hence,

\[
w(x, y) = \sum_{m} \sum_{n} \frac{p_{mn}}{D} \cos \frac{m \pi x}{a} \cos \frac{n \pi y}{b} \cos \frac{m \pi x}{a} \cos \frac{n \pi y}{b} \quad \ldots (27)
\]

The bending moment distributions are

\[
M_{xx} = -D \left( \frac{\partial^2 w}{\partial x^2} + \mu \frac{\partial^2 w}{\partial y^2} \right) \quad \ldots (28)
\]

\[
M_{yy} = -D \left( \frac{\partial^2 w}{\partial y^2} + \mu \frac{\partial^2 w}{\partial x^2} \right) \quad \ldots (29)
\]

\[
M_{xy} = -D(1 - \mu) \frac{\partial^2 w}{\partial x \partial y} \quad \ldots (30)
\]

Using the results for \( w(x, y) \) we have

\[
M_{xx} = \sum_{m} \sum_{n} \left( \left( \frac{m \pi}{a} \right)^2 \left( \frac{n \pi}{b} \right)^2 + \mu \left( \frac{m \pi}{a} \right)^2 \left( \frac{n \pi}{b} \right)^2 \right) p_{mn} \cos \frac{m \pi x}{a} \cos \frac{n \pi y}{b} \quad \ldots (31)
\]

\[
M_{yy} = \sum_{m} \sum_{n} \left( \left( \frac{m \pi}{a} \right)^2 \left( \frac{n \pi}{b} \right)^2 + \mu \left( \frac{m \pi}{a} \right)^2 \left( \frac{n \pi}{b} \right)^2 \right) p_{mn} \cos \frac{m \pi x}{a} \cos \frac{n \pi y}{b} \quad \ldots (32)
\]

\[
M_{xy} = -(1 - \mu) \sum_{m} \sum_{n} \left( \left( \frac{m \pi}{a} \right)^2 \left( \frac{n \pi}{b} \right)^2 + \left( \frac{m \pi}{a} \right)^2 \left( \frac{n \pi}{b} \right)^2 \right) p_{mn} \sin \frac{m \pi x}{a} \sin \frac{n \pi y}{b} \quad \ldots (33)
\]

4. Results

4.1 Point load \( P_1 \) at \( (x_1, y_1) \) where \(-\frac{a}{2} < x_1 < \frac{a}{2}; \quad -\frac{b}{2} < y_1 < \frac{b}{2}\).

The results for \( p_{mn}, w_{mn}, w(x, y), M_{xx}, \) and \( M_{xy} \) for Kirchhoff plate resting on Winkler foundation under point load \( P_1 \) acting at an arbitrary point \( (x_1, y_1) \) on the plate domain is obtained using Dirac delta function theory in the general solutions for arbitrary transverse load distribution. Thus,

\[
p_{mn} = \frac{4}{ab} \int_{-a/2}^{a/2} \int_{-b/2}^{b/2} P_1 \delta(x = x_1; y = y_1) \cos \frac{m \pi x}{a} \cos \frac{n \pi y}{b} \, dx \, dy \quad \ldots (34)
\]

\[
p_{mn} = \frac{4}{ab} P_1 \int_{-a/2}^{a/2} \int_{-b/2}^{b/2} \delta(x = x_1; y = y_1) \cos \frac{m \pi x}{a} \cos \frac{n \pi y}{b} \, dx \, dy \quad \ldots (35)
\]

\[
p_{mn} = \frac{4P_1}{ab} \cos \frac{m \pi x_1}{a} \cos \frac{n \pi y_1}{b} \quad \ldots (36)
\]

\[
w_{mn} = \frac{4P_1}{ab} \cos \frac{m \pi x_1}{a} \cos \frac{n \pi y_1}{b} \quad \ldots (37)
\]

\[
w_{mn} = \frac{4P_1}{ab} \cos \frac{m \pi x_1}{a} \cos \frac{n \pi y_1}{b} \cos \frac{m \pi x}{a} \cos \frac{n \pi y}{b} \quad \ldots (38)
\]
For a point load $P_1$ acting at the centre of the Kirchhoff plate on Winkler foundation, $x_1 = 0, y_1 = 0$, and the maximum deflection and bending moments which occur at the centre are given by:

$$w_{\text{max}} = \frac{4P_1}{a^2D} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{\pi^2 m^2 + \pi^2 n^2 + \frac{k a^4}{D}}$$

$$M_{xx_{\text{max}}} = \frac{4P_1}{a^2b} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left( \frac{\bar{m}^2}{a^2} + \frac{\bar{n}^2}{b^2} \right) \cos \frac{\bar{m} \pi x}{a} \cos \frac{\bar{n} \pi y}{b}$$

$$M_{yy_{\text{max}}} = \frac{4P_1}{a^2b} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left( \frac{\bar{m}^2}{a^2} + \frac{\bar{n}^2}{b^2} \right) \cos \frac{\bar{m} \pi x}{a} \cos \frac{\bar{n} \pi y}{b}$$

$$M_{xx} = \frac{4P_1}{ab} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left( \frac{\bar{m}^2}{a^2} + \frac{\bar{n}^2}{b^2} \right) \cos \frac{\bar{m} \pi x}{a} \cos \frac{\bar{n} \pi y}{b}$$

$$M_{yy} = \frac{4P_1}{ab} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left( \frac{\bar{m}^2}{a^2} + \frac{\bar{n}^2}{b^2} \right) \cos \frac{\bar{m} \pi x}{a} \cos \frac{\bar{n} \pi y}{b}$$

$$M_{xx_{\text{max}}} = \frac{4P_1}{ab} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left( \frac{\bar{m}^2}{a^2} + \frac{\bar{n}^2}{b^2} \right) \cos \frac{\bar{m} \pi x}{a} \cos \frac{\bar{n} \pi y}{b}$$

$$M_{yy_{\text{max}}} = \frac{4P_1}{ab} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left( \frac{\bar{m}^2}{a^2} + \frac{\bar{n}^2}{b^2} \right) \cos \frac{\bar{m} \pi x}{a} \cos \frac{\bar{n} \pi y}{b}$$

4.2 Results for transverse load distribution as

$$p(x,y) = p_0 \cos \frac{\pi x}{a} \cos \frac{\pi y}{b}$$

$\frac{a}{2} < x < \frac{a}{2}$

$\frac{b}{2} < y < \frac{b}{2}$

Here,

$$P_{\text{mn}} = 4 \int_{-a/2}^{a/2} \int_{-b/2}^{b/2} p_0 \cos \frac{\pi x}{a} \cos \frac{\pi y}{b} \cos \bar{m} \pi x a \cos \bar{n} \pi y b \, dx dy$$

$$p_{\text{mn}} = \frac{4P_0}{a^2b} \int_{-\bar{m}}^{\bar{m}} \int_{-\bar{n}}^{\bar{n}} \cos \frac{\pi x}{a} \cos \frac{\pi y}{b} \, dx \, dy$$

$$P_{\text{mn}} = \frac{4P_0}{a^2b} \int_{-\bar{m}}^{\bar{m}} \int_{-\bar{n}}^{\bar{n}} \cos \frac{\pi x}{a} \cos \frac{\pi y}{b} \, dx \, dy$$

$$p_{\text{mn}} = 0$$

where $\bar{m} = 1, \bar{n} = 1$
\[ w(x, y) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \frac{p_0 \cos \frac{m\pi x}{a} \cos \frac{n\pi y}{b}}{D \left( \left( \frac{m\pi}{a} \right)^2 + \left( \frac{n\pi}{b} \right)^2 + \frac{k}{D} \right)} \]  

(61)

\[ w(x, y) = \frac{p_0}{D} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \frac{1}{\pi^4 \left( \left( \frac{m\pi}{a} \right)^2 + \left( \frac{n\pi}{b} \right)^2 + \frac{k}{D} \right)} \]  

(62)

\[ w(x = 0, y = 0) = \frac{p_0}{D} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \frac{1}{\pi^4 \left( \left( \frac{m\pi}{a} \right)^2 + \left( \frac{n\pi}{b} \right)^2 + \frac{k}{D} \right)} \]  

(63)

\[ w(0, 0) = \frac{p_0}{D} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \frac{1}{\pi^4 \left( \left( \frac{m\pi}{a} \right)^2 + \left( \frac{n\pi}{b} \right)^2 + \frac{k}{D} \right)} \]  

(64)

\[ M_{xx} = \left( \frac{\pi^2}{a^2} + \mu \frac{\pi^2}{b^2} \right) \frac{p_0 \cos \frac{m\pi x}{a} \cos \frac{n\pi y}{b}}{a} \]  

(65)

\[ M_{yy} = \left( \frac{n^2}{b^2} + \mu \frac{n^2}{a^2} \right) \frac{p_0 \cos \frac{m\pi x}{a} \cos \frac{n\pi y}{b}}{a} \]  

(66)

\[ M_{xx} = -(1 - \mu) \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \frac{\sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}}{\left( \frac{m^2\pi^2}{a^2} + \frac{n^2\pi^2}{b^2} \right)^2 + \frac{ka^4}{D}} \]  

(67)

\[ M_{yy} = -(1 - \mu) \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \frac{\sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}}{\left( \frac{m^2\pi^2}{a^2} + \frac{n^2\pi^2}{b^2} \right)^2 + \frac{ka^4}{D}} \]  

(68)

The maximum values of the deflection and bending moments occur at the plate centre \( x = 0, y = 0 \) and are given by:

\[ w_{\text{max}} = \frac{p_0 a^2}{D} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \frac{1}{\pi^4 \left( \left( \frac{m\pi}{a} \right)^2 + \left( \frac{n\pi}{b} \right)^2 \right)^2 + \frac{ka^4}{D}} \]  

(73)

\[ M_{xx \text{ max}} = M_{xx}(0, 0) = \left( \frac{\pi^2}{a^2} + \mu \frac{\pi^2}{b^2} \right) \frac{p_0}{a} \]  

(74)

\[ M_{yy \text{ max}} = M_{yy}(0, 0) = \left( \frac{n^2}{b^2} + \mu \frac{n^2}{a^2} \right) \frac{p_0}{a} \]  

(75)

\[ M_{xx \text{ max}} = \left( \frac{\pi^2}{a^2} + \mu \frac{\pi^2}{b^2} \right) \frac{p_0}{a} \]  

(76)

\[ M_{yy \text{ max}} = \left( \frac{n^2}{b^2} + \mu \frac{n^2}{a^2} \right) \frac{p_0}{a} \]  

(77)

\[ M_{xx \text{ max}} = \frac{\pi^2}{a^2} \left( 1 + \mu \right) \frac{p_0}{a^2} \]  

(78)

\[ M_{yy \text{ max}} = \frac{\pi^2}{b^2} \left( 1 + \mu \right) \frac{p_0}{b^2} \]  

(79)

\[ M_{xx \text{ max}} = -\left( 1 + \mu \right) \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \frac{\sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}}{\left( \frac{m^2\pi^2}{a^2} + \frac{n^2\pi^2}{b^2} \right)^2 + \frac{ka^4}{D}} \]  

(80)

4.3 Results for uniformly distributed load \( p(x, y) = p \) over the entire plate domain

For the case of uniformly distributed transverse load of intensity \( p \) over the entire plate region, the double Fourier cosine series coefficient \( p_{mn} \) is found as:

\[ p_{mn} = \frac{1}{ab} \int_{-b/2}^{b/2} \int_{-a/2}^{a/2} p \cos \frac{m\pi x}{a} \cos \frac{n\pi y}{b} \, dx \, dy \]  

(81)

\[ p_{mn} = \frac{1}{ab} \int_{-b/2}^{b/2} \int_{-a/2}^{a/2} p \cos \frac{m\pi x}{a} \cos \frac{n\pi y}{b} \, dx \, dy \]  

(82)

\[ p_{mn} = \frac{1}{\sin m\pi a} \left[ \frac{b}{\pi} \sin \frac{m\pi y}{b} \right]_{-b/2}^{b/2} \]  

(83)

\[ p_{mn} = \frac{1}{\sin m\pi a} \left[ \frac{b}{\pi} \sin \frac{m\pi y}{b} \right]_{-b/2}^{b/2} \]  

(84)

\[ p_{mn} = \frac{4}{ab} \frac{p_0}{a^2} \left( \frac{m\pi}{a} \sin \frac{m\pi}{a} - \sin \frac{m\pi}{a} \right) \left( \frac{m\pi}{a} \sin \frac{m\pi}{a} + \sin \frac{m\pi}{a} \right) \]  

(85)

\[ p_{mn} = \frac{4}{ab} \frac{p_0}{a^2} \left( \frac{m\pi}{a} \sin \frac{m\pi}{a} - \sin \frac{m\pi}{a} \right) \left( \frac{m\pi}{a} \sin \frac{m\pi}{a} + \sin \frac{m\pi}{a} \right) \]  

(86)

Then,
\( w_{\text{max}} \) = \frac{16 \rho}{D \pi a b^4 \pi^2} \left( \frac{\tilde{m}}{a} + \frac{\tilde{n}}{b} \right)^2 + k \left( \frac{D}{a} \right)^2 \) \( \ldots (87) \)

\( w(x, y) = \sum_{m} \sum_{n} \frac{16 \rho}{D \pi a b^4 \pi^2} \left( \frac{\tilde{m}}{a} \frac{\tilde{n}}{b} \right)^2 + k \left( \frac{D}{a} \right)^2 \) \( \ldots (88) \)

\( w(x, y) = \sum_{m} \sum_{n} \frac{16 \rho}{D \pi a b^4 \pi^2} \left( \frac{\tilde{m}}{a} \frac{\tilde{n}}{b} \right)^2 + k \left( \frac{D}{a} \right)^2 \) \( \ldots (89) \)

\( M_{xx} = \frac{16 \rho}{D \pi a b^4 \pi^2} \sum_{m} \sum_{n} \cos \left( \frac{\tilde{m}}{a} \right) \cos \left( \frac{\tilde{n}}{b} \right) \left( \frac{\tilde{m}}{a} \frac{\tilde{n}}{b} \right)^2 + k \left( \frac{D}{a} \right)^2 \) \( \ldots (90) \)

\( M_{yy} = \frac{16 \rho}{D \pi a b^4 \pi^2} \sum_{m} \sum_{n} \cos \left( \frac{\tilde{m}}{a} \right) \cos \left( \frac{\tilde{n}}{b} \right) \left( \frac{\tilde{m}}{a} \frac{\tilde{n}}{b} \right)^2 + k \left( \frac{D}{a} \right)^2 \) \( \ldots (91) \)

\( M_{xy} = (1 - \mu) \sum_{m} \sum_{n} \sin \left( \frac{\tilde{m}}{a} \right) \sin \left( \frac{\tilde{n}}{b} \right) \left( \frac{\tilde{m}}{a} \frac{\tilde{n}}{b} \right)^2 + k \left( \frac{D}{a} \right)^2 \) \( \ldots (92) \)

5. Discussion

The method of double Fourier cosine series has been effectively used in this work to solve the flexural problem of simply supported Kirchhoff plate resting on Winkler foundation. The problem is described mathematically as a boundary value problem given by Equations (5) and (8 – 11) where Equations (8 – 11) represent the boundary conditions for the choice of origin at the centre, and Equation (5) is the domain governing partial differential equation. Double Fourier cosine series of the form given by Equation (12) was shown to identically satisfy the boundary conditions, and for the transverse load distribution given by Equation (13), the boundary value problem was shown to decompose to an algebraic problem. Equation (23) in terms of \( w_{\text{max}} \), the unknown parameters of the double Fourier cosine series.

The algebraic eigenvalue problem was solved to obtain the unknown generalised displacement parameter \( w_{\text{max}} \) of the double Fourier cosine series as Equation (26). The unknown deflection was thus obtained for general arbitrary distribution of transverse load as Equation (27). The solutions for bending and twisting moments were found using the bending moment – curvature expressions, as Equations (31 – 33).

For the specific case of point load \( P \), at an arbitrary point on the plate domain, the deflection function was obtained as Equation (39) and the bending moments were found as Equations (40) and (41). For point load applied at the centre for square Kirchhoff plates on Winkler foundation the maximum values for deflection and bending moments were obtained as Equation (53), (54) and (55).

The solution of the flexural problem of simply supported rectangular Kirchhoff plate on Winkler foundation for the case of sinusoidal load distribution was obtained as Equation (61) for deflection and Equations (65 – 67) for bending and twisting moments. In that case, the maximum values of deflection and bending moments were found to occur at the plate centre, and are given by Equations (70), (71) and (72). For square Kirchhoff plates on Winkler’s foundation, the maximum deflection occurred at the centre and was found as Equation (75); the maximum bending moment occurred at the centre and was found as Equation (78). The maximum twisting moment was found as Equation (80).

For the case of simply supported Kirchhoff plate on Winkler foundation, the deflection and bending moment expressions were found as Equations (89), (90) and (91). The maximum values of the deflection and bending moments were found in compliance with symmetry of the problem to occur of the plate centre and were found as Equations (93), (94) and (95).

The double Fourier cosine series method give solutions for simply supported square Kirchhoff plate on Winkler foundation for the case of uniformly distributed transverse load for various values of the non-dimensional Winkler parameter \( KND \) varying from \( KND = 0 \) to \( KND = 5 \) as shown in Table 1. The solutions for maximum deflection and maximum bending and twisting moments obtained for simply supported square Kirchhoff plate on Winkler foundation for the case of sinusoidal load for different values of non-dimensional Winkler parameter \( KND = \frac{kD}{a^4} \) varying from \( KND = 0 \) to \( KND = 7 \) were calculated and tabulated in Table 2.

Table 1 shows that the maximum deflections and maximum bending moments which occur at the plate centre decrease as the elastic stiffness of the Winkler foundation defined by the non-
dimensional Winkler parameter increases. From Table 2, it is similarly observed that with increase in the elastic stiffness of the soil as measured by the non-dimensional Winkler parameter $K_{ND}$, the maximum deflections, bending and twisting moments reduce. It is also observed that the solutions obtained for deflection, and bending moments for all cases of load considered were double trigonometric (cosine) series of infinite terms. The double cosine series obtained for deflection were rapidly convergent for sinusoidal and uniformly distributed loads, but less rapidly convergent for point load, obviously due to the singular property of point loads. The series obtained for bending moments were less rapidly convergent for sinusoidal and uniformly distributed loads. The rapid convergence of the double Fourier cosine series obtained for the deflections for distributed loads led to sufficiently accurate results with only a few terms of the series. However, more terms of the series for bending moments were used for satisfactorily convergent results.

Table 1: Galerkin solution for deflection and bending moment coefficients for simply supported Kirchhoff plate on Winkler foundation for uniform load on the plate and square plates

<table>
<thead>
<tr>
<th>$K_{ND}$</th>
<th>$w_{max} \times 10^{-3} \frac{pa^4}{D}$</th>
<th>$M_{xx, max} \times 10^{-2} pa^2$</th>
<th>$M_{yy, max} \times 10^{-2} pa^2$</th>
<th>$M_{sy, max} \times 10^{-2} pa^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>4.062</td>
<td>4.790</td>
<td>4.790</td>
<td>2.943</td>
</tr>
<tr>
<td>1</td>
<td>4.053</td>
<td>4.809</td>
<td>4.809</td>
<td>2.456</td>
</tr>
<tr>
<td>3</td>
<td>3.348</td>
<td>3.910</td>
<td>3.910</td>
<td>1.181</td>
</tr>
<tr>
<td>5</td>
<td>1.507</td>
<td>1.575</td>
<td>1.575</td>
<td></td>
</tr>
</tbody>
</table>

Table 2: Galerkin solution for maximum deflections and bending moment coefficients for simply supported square Kirchhoff plate on Winkler foundations for sinusoidal load $p(x, y) = p_0 \cos \frac{\pi x}{a} \cos \frac{\pi y}{b}$, $\mu = 0.30$

<table>
<thead>
<tr>
<th>$K_{ND}$ = $\left( \frac{ka^4}{D} \right)^{1/4}$</th>
<th>$w_{max} \times 10^{-3} \frac{pa^4}{D}$</th>
<th>$M_{xx} \times p_0 a^2$</th>
<th>$M_{yy} \times p_0 a^2$</th>
<th>$M_{xy} \times p_0 a^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2.5665</td>
<td>0.03293</td>
<td>0.03293</td>
<td>0.001797</td>
</tr>
<tr>
<td>1</td>
<td>2.5599 ≈ 2.56</td>
<td>0.03285</td>
<td>0.03285</td>
<td>0.001792</td>
</tr>
<tr>
<td>3</td>
<td>2.1248</td>
<td>0.02726</td>
<td>0.02726</td>
<td>0.001487</td>
</tr>
<tr>
<td>5</td>
<td>0.98557</td>
<td>0.01265</td>
<td>0.01265</td>
<td>0.0006899</td>
</tr>
<tr>
<td>7</td>
<td>0.35834</td>
<td>0.0045977</td>
<td>0.0045977</td>
<td>0.00025084 ≈ 2.508 \times 10^{-4}</td>
</tr>
</tbody>
</table>

6. Conclusions
The conclusions of the work are as follows:

1. The double Fourier cosine series method simplifies the problem of flexure of simply supported Kirchhoff plate resting on Winkler foundation to an algebraic problem in terms of the unknown generalised deflection parameters $w_{max}$ of the double Fourier cosine series of the deflection function.
2. The method gave analytically closed form solutions for the deflections, bending and twisting moments for the simply supported Kirchhoff plate on Winkler foundation problem under transverse distributed load.
3. As the elastic stiffness of the Winkler foundation increased, the maximum deflections and maximum bending moments reduced.
4. The Dirichlet boundary conditions associated with the simply supported edges simplified the use of the double Fourier cosine series method.
5. The double Fourier cosine series expressions obtained for the deflections were more rapidly convergent than the expressions obtained for the bending and twisting moments.
6. The double Fourier cosine series expression obtained for the deflections and bending moments in the case of point load acting on the plate on Winkler foundation were very slow in convergence due to the singularity property of the point load.

References